



מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Energy-conserving theory of strongly nonlinear plasma wakefields

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Paper: A. Golovanov et al. *PRL* **130**, 105001 (2023)

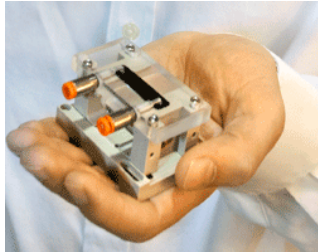
Stanford linear acceleration center (SLAC)



50 GeV at 3 km

Acceleration gradient ~ 100 MeV/m

Plasma accelerators

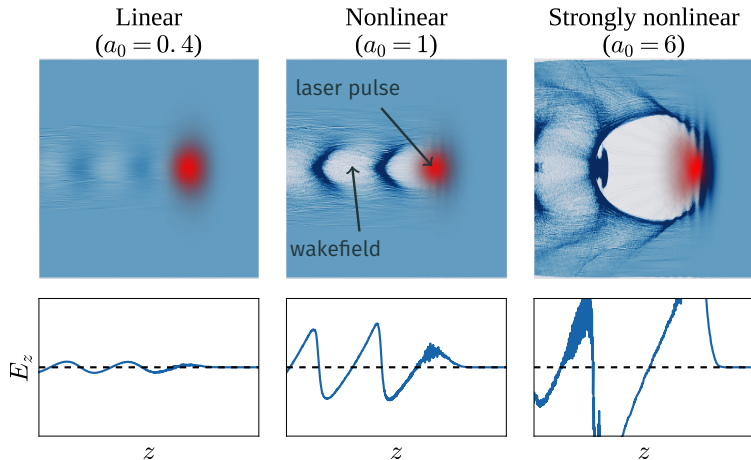


7.8 GeV at 20 cm.

Acceleration gradient ~ 50 GeV/m

Gonsalves *et al.* PRL **122**, 084801 (2019).

Wakefield in plasma



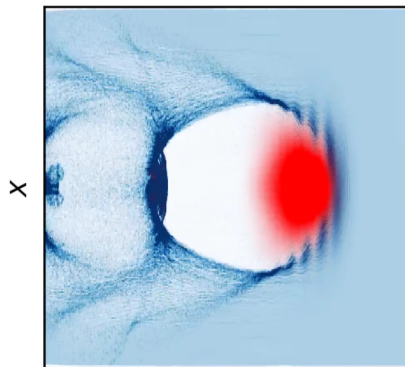
$$a_0 = \frac{eE_L}{mc\omega_L}$$

is the unitless laser amplitude

Strongly nonlinear (“bubble” or “blowout”) regime of wakefield

Laser driver ($a_0 \gg 1$)

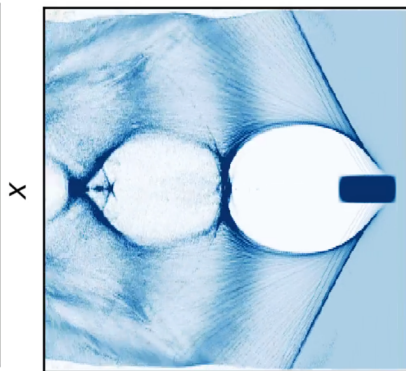
n_e



$z-ct$

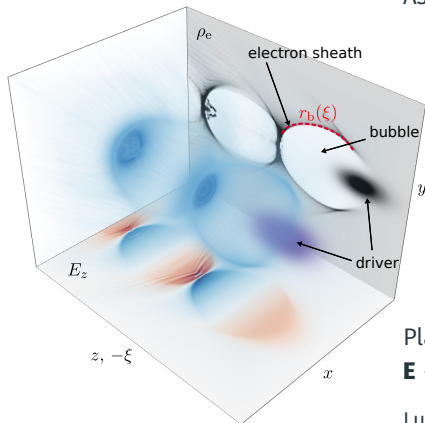
Electron driver ($n_B \gg n_p$)

n_e



$z-ct$

Phenomenological models of plasma wakefield



Assumptions

- Axial symmetry, coordinates (r, z)
- Quasi-static approximation:
 $f(t, z, r) = f(\xi, r), \quad \xi = t - z.$
- The plasma bubble has a border $r_b(\xi)$. For $r < r_b$, there are no plasma electrons; $r \geq r_b$ lies the electron sheath
- Ions are immobile

Plasma units: $t \rightarrow \omega_p t, \mathbf{r} \rightarrow k_p \mathbf{r},$
 $\mathbf{E} \rightarrow e\mathbf{E}/mc\omega_p,$ etc.

Lu et al. *Phys. Plasmas* **13**, 056709 (2006)

Plasma boundary equation

The boundary of the bubble is described by the equation

$$A(r_b) \frac{d^2 r_b}{d\xi^2} + B(r_b) \left(\frac{dr_b}{d\xi} \right)^2 + C(r_b) = \lambda(\xi, r_b),$$

where

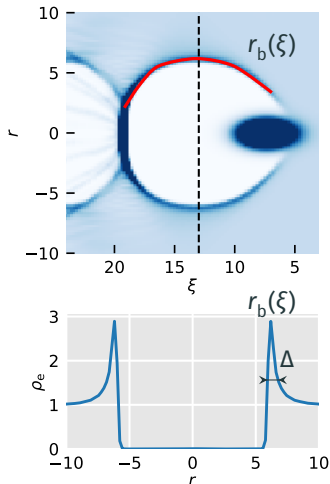
$$A(r_b) = r_b \left(1 + \frac{r_b^2}{4} + \frac{3r_b \Delta}{4} \right)$$

$$B(r_b) = \frac{r_b^2}{2} \left(1 + \frac{\Delta}{r_b} \right)$$

$$C(r_b) = \frac{r_b^2}{4} \frac{1 + (1 + r_b \Delta/2)^2}{(1 + r_b \Delta/2)^2}$$

$$\lambda(\xi, r_b) = - \int_0^{r_b} \rho_e(\xi, r') r' dr'$$

$$E_z(\xi) = \frac{r_b + \Delta}{2} \frac{dr_b}{d\xi}$$



Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

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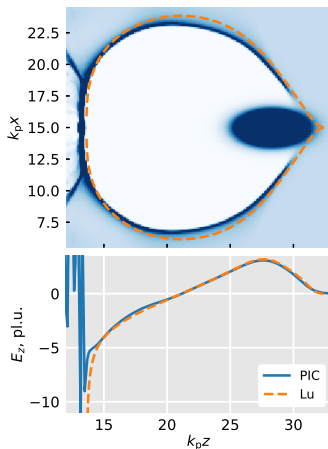
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EM field (wakefield)

$$\frac{\partial W_{EM}}{\partial t} + \nabla \cdot \mathbf{S}_{EM} = -\mathbf{j} \cdot \mathbf{E}$$

$$W_{EM} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2} \text{ EM energy density}$$

$$\mathbf{S}_{EM} = \mathbf{E} \times \mathbf{B} \text{ Poynting vector}$$

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Plasma electrons

$$\frac{\partial W_e}{\partial t} + \nabla \cdot \mathbf{S}_e = \mathbf{j}_e \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

$$W_e = n_e \overline{(\gamma - 1)} \text{ electron energy density}$$

$$\mathbf{S}_e = n_e \mathbf{v} (\gamma - 1) \text{ energy density current}$$

$$\langle \mathbf{a}^2 \rangle \text{ ponderomotive potential}$$

Energy conservation in a plasma wakefield

EM field (wakefield)

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Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

$$W = W_{EM} + W_e \quad \mathbf{S} = \mathbf{S}_{EM} + \mathbf{S}_e$$

Currents $\mathbf{j} = \mathbf{j}_e + \mathbf{j}_B$ belong to plasma electrons (\mathbf{j}_e) or to external bunches (\mathbf{j}_B).

Quasistatic approximation

Total wake energy

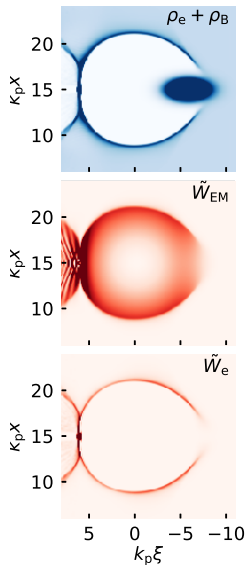
$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \gamma^{-1}$$

Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_{\perp} \cdot \mathbf{S}_{\perp} = -\rho_B E_z + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \gamma^{-1}$$

$\tilde{W} = W - S_z$ is the quasi-energy density.

Ultrarelativistic bunches: $\mathbf{j}_B \approx \rho_B \mathbf{z}_0$



Quasistatic approximation

Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \gamma^{-1}$$

Quasistatic approximation ($\xi = t - z$)

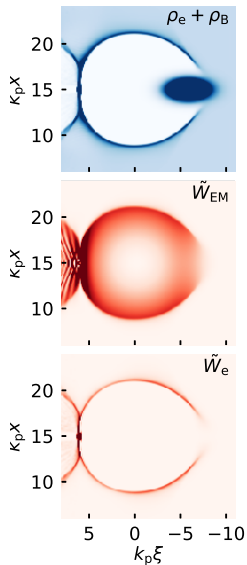
$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_{\perp} \cdot \mathbf{S}_{\perp} = -\rho_B E_z + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \gamma^{-1}$$

$\tilde{W} = W - S_z$ is the quasi-energy density.

Ultrarelativistic bunches: $\mathbf{j}_B \approx \rho_B \mathbf{z}_0$

$$\tilde{W}_{EM} = \frac{1}{2} [(\nabla \psi_w)^2 + B_z^2] \quad \tilde{W}_e = n_e \overline{(\gamma - 1)(1 - v_z)}$$

$\psi_w = \varphi - A_z$ is the wakefield potential.



Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

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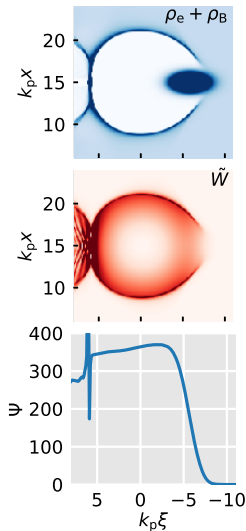
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Integrating over the transverse plane

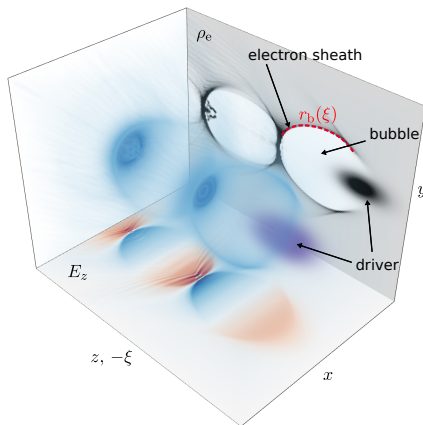
$$\frac{d\Psi}{d\xi} = - \int \rho_B E_z d^2 \mathbf{r}_{\perp} + \frac{1}{2} \int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}} d^2 \mathbf{r}_{\perp}$$

$$\Psi(\xi) = \int \tilde{W} d^2 \mathbf{r}_{\perp} \quad (\text{slice quasi-energy})$$

Behind the driver, no witness bunches: $\Psi(\xi) = \text{const.}$



Model of the bubble



- Axial symmetry (r, z), only E_z, E_r, B_ϕ components.
- The bubble has a boundary, $r_b(\xi)$.
- Inside the boundary, no plasma electrons.
- The electron sheath on the boundary is infinitely thin.

$$\Psi = \Psi_{EM} + \Psi_e$$

EM slice quasi-energy

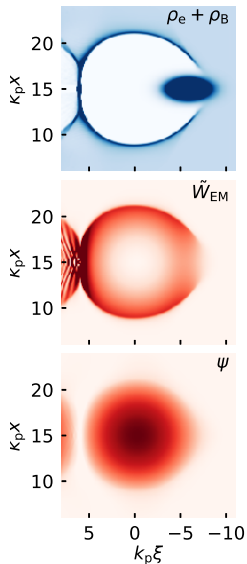
$$\tilde{W}_{EM} = \frac{(\nabla\psi_w)^2}{2} \quad \Psi_{EM}(\xi) = \pi \int_0^{r_b(\xi)} (\nabla\psi_w)^2 r dr$$

The wakefield potential inside the bubble is

$$\psi_w(\xi, r) = \frac{r_b^2(\xi) - r^2}{4}$$

Therefore

$$\Psi_{EM}(\xi) = \frac{\pi r_b^4}{16} \left[1 + 2 \left(\frac{dr_b}{d\xi} \right)^2 \right]$$



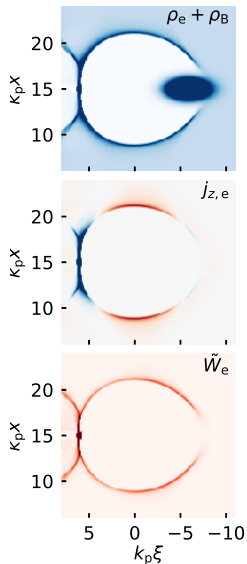
Electron slice quasi-energy

We define the electron sheath as

$$j_{z,e} = j_0(\xi) r_b \delta(r - r_b)$$

Then,

$$\Psi_e(\xi) = 2\pi \int_{r_b-0}^{r_b+0} \tilde{W}_e r dr = \frac{\pi}{2} r_b^2 \left(\frac{dr_b}{d\xi} \right)^2$$



Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16} \left[r_b^2 + (2r_b^2 + 8) \left(\frac{dr_b}{d\xi} \right)^2 \right], \quad \frac{d\Psi}{d\xi} = -\pi r_b \frac{dr_b}{d\xi} \int_0^{r_b} \rho_B r dr$$

Equation for the bubble boundary

Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16} \left[r_b^2 + (2r_b^2 + 8) \left(\frac{dr_b}{d\xi} \right)^2 \right], \quad \frac{d\Psi}{d\xi} = -\pi r_b \frac{dr_b}{d\xi} \int_0^{r_b} \rho_B r dr$$

And we get the equation

$$\left(\frac{r_b^3}{4} + r_b \right) \frac{d^2 r_b}{d\xi^2} + \left(\frac{r_b^2}{2} + 1 \right) \left(\frac{dr_b}{d\xi} \right)^2 + \frac{r_b^2}{2} = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r dr$$

Red terms — EM energy, blue — plasma electrons energy.

Bubble equation = energy conservation

Comparison of equations

$$A(r_b) \frac{d^2 r_b}{d\xi^2} + B(r_b) \left(\frac{dr_b}{d\xi} \right)^2 + C(r_b) = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r dr$$

Energy conservation

$$A = \frac{r_b^3}{4} + r_b \quad B = \frac{r_b^2}{2} + 1$$
$$C = \frac{r_b^2}{4} \quad E_z = \frac{r_b}{2} \frac{dr_b}{d\xi}$$

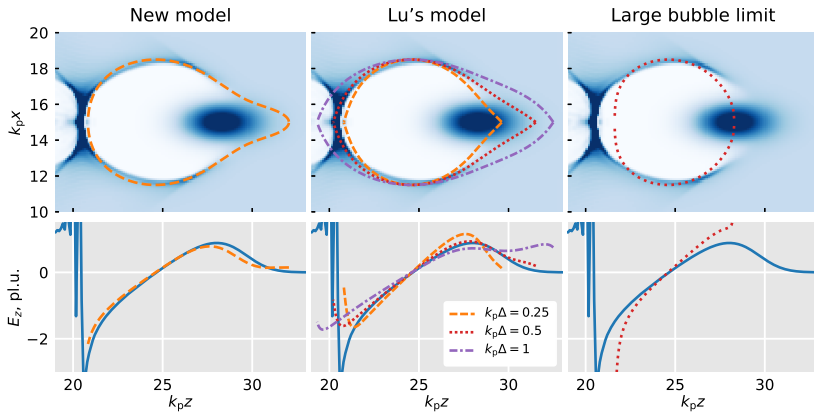
Lu's model

$$A = \frac{r_b^3}{4} + r_b + \frac{3}{4} r_b^2 \Delta, \quad B = \frac{r_b^2}{2} + \frac{r_b \Delta}{2},$$
$$C = \frac{r_b^2}{4} \left[1 + \left(1 + \frac{r_b \Delta}{2} \right)^{-2} \right], \quad E_z = \frac{r_b + \Delta}{2} \frac{dr_b}{d\xi}.$$

Does not correspond to energy conservation

For large bubbles $r_b \gg 1$, we get the same equation.

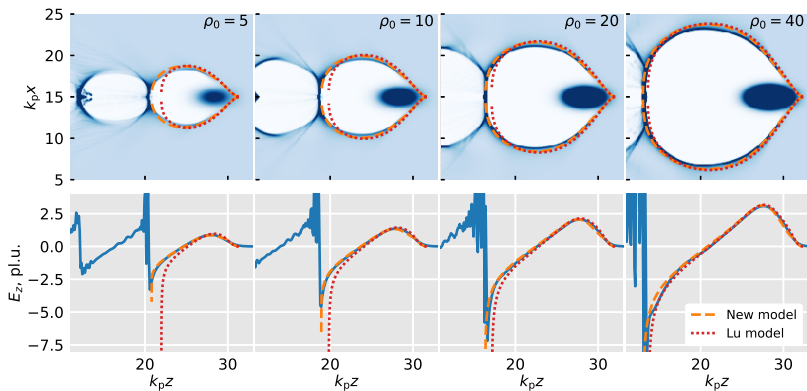
Comparison of models



Bubble excitation

Self-consistent excitation of the bubble by an electron driver.

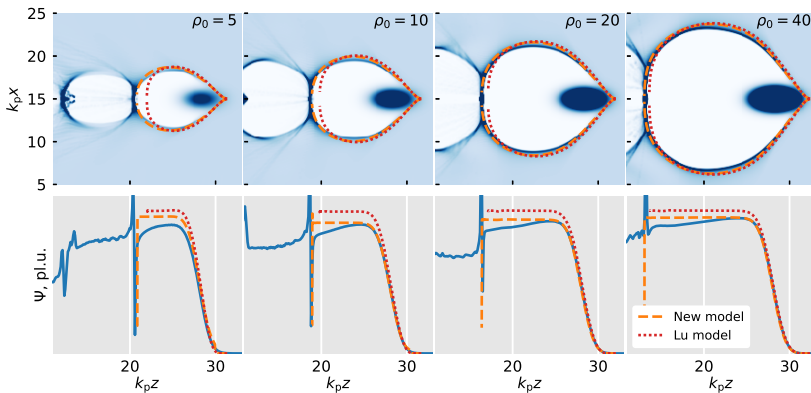
Golovanov et al. *PPCF* **63**, 085004 (2021).



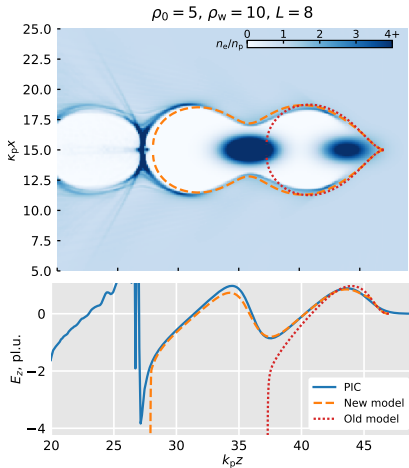
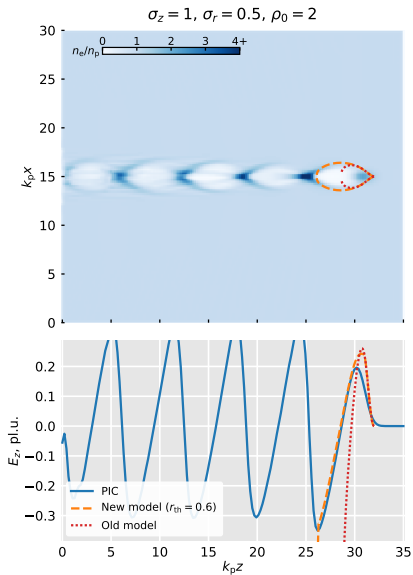
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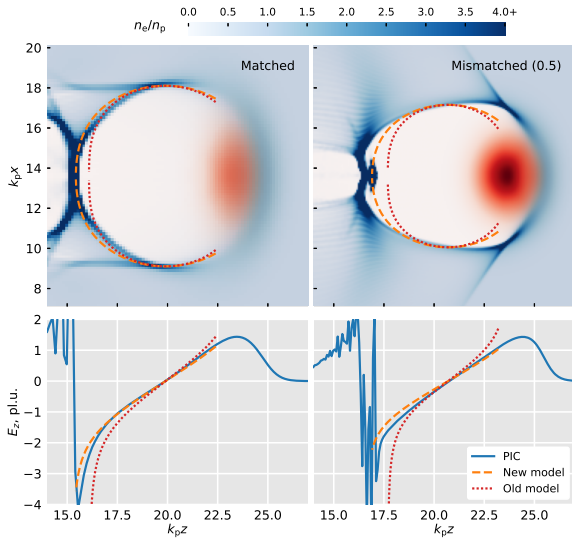
Golovanov et al. *PPCF* **63**, 085004 (2021).



Other examples



Laser driver



Thank you for your attention!

Paper: A. Golovanov et al. *PRL* 130, 105001 (2023)